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Multiresonant control of two-dimensional dynamical systems

Ido Barth and Lazar Friedland

Racah Institute of physics, Hebrew University of Jerusalem

Abstract

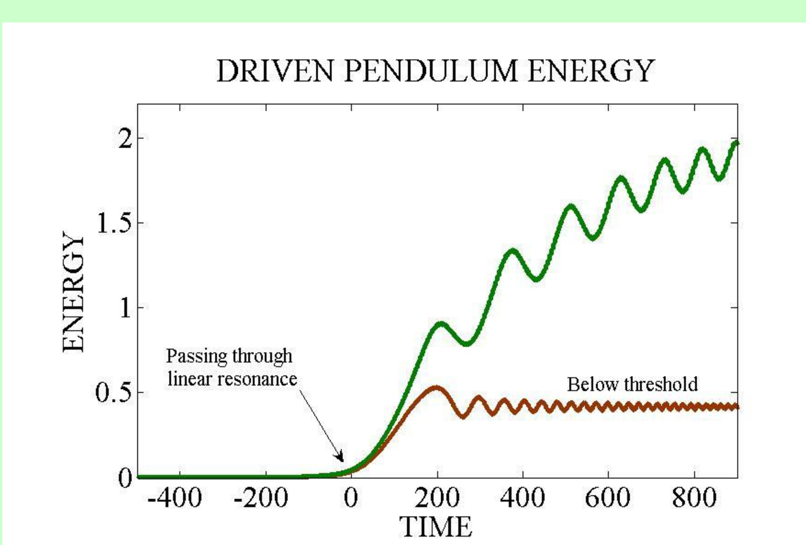
Autoresonance is a nonlinear phenomenon characterized by continuing phase-locking between a dynamical system and external oscillatory perturbations despite slow variation of system's parameters. Multi-frequency autoresonance in 2D driven dynamical systems is discussed. The problem of doubly resonance trapping by simultaneous passage through non-degenerate linear resonance and the associate threshold phenomenon are examined. Classification of systems enabling AR solution in the vicinity of equilibrium is described in terms of the Hessian matrix of the unperturbed system. Examples autoresonant excitation of periodic trajectories in several systems are presented.

Introduction

The single frequency autoresonace control is currently well understood. Simple example for 1D autoresonance is the driven pendulum,

$$\ddot{q} + \sin q = \varepsilon \cos\varphi_d$$

Where the drive phase is $\varphi = \int \omega_d dt$ while the drive frequency $\omega(t) = 1 - \alpha t$ passes through linear resonance at t=0. Phase-locking is preserved and the energy grows slowly with small oscillations. There is a threshold phenomenon with critical drive amplitude scaling as $\varepsilon^{cr} \sim \alpha^{3/4}$.



Our system

Consider two dimensions driven dynamic system:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \underbrace{V(q_1, q_2)}_{2D\ potential} + \underbrace{\varepsilon_1 q_1 \cos\varphi_1 + \varepsilon_2 q_2 \cos\varphi_2}_{drivers}$$

$$\varphi_i = \int \omega_i dt \qquad i = 1,2$$

$$\omega_i(t) = \nu_i + \alpha_i t,$$

$$\varepsilon_i << 1.$$

ν_i are the linear frequencies of the unperturbed system, So the drivers frequencies pass through linear resonances $\omega_i(t) = \nu_i$ at t=0. A continuing *double phase-locking* is establish and preserved due to nonlinearity .

Theory & Results

Action-Angle representation

We write the Hamiltonian in Action-Angle variables near the equilibrium. In non-degenerated systems $(\nu_1 \neq \nu_2)$ this can be conveniently done via the canonical perturbation theory.

$$H = \underbrace{\nu_1 I_1 + \nu_2 I_2}_{Linear\ term} + \underbrace{\frac{1}{2} a I_1^2 + b I_1 I_2 + \frac{1}{2} c I_2^2 + O\left(I^3\right)}_{non\ Linear\ term} + \underbrace{f\left(\mathbf{I}, \boldsymbol{\theta}, t\right)}_{Drivers}$$

Double resonance approximation:

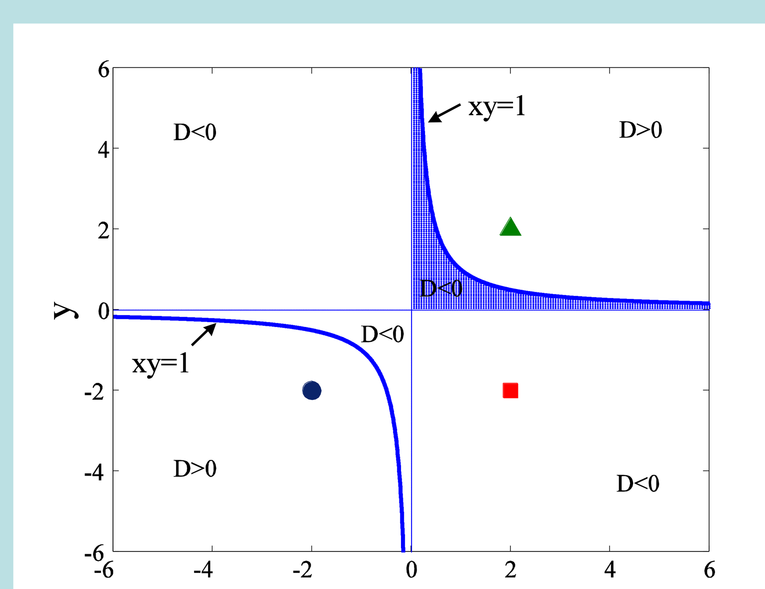
$$f = \varepsilon_1 \sqrt{\frac{I_1}{2\nu_1}} \cos\psi_1 + \varepsilon_2 \sqrt{\frac{I_2}{2\nu_2}} \cos\psi_2$$

Phase mismatches: $\psi_i = \theta_i - \varphi_i$

Evolution equations: $$\dot{\psi}_i = \frac{\partial H}{\partial I_i} - \omega_i(t) \approx 0$$

Resonance condition

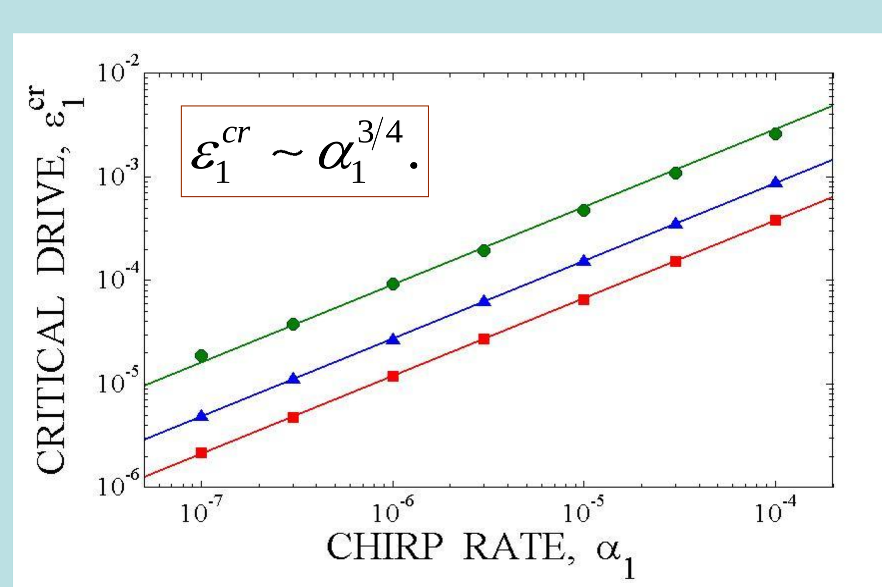
Classification approach



$$x = \frac{a}{b} \qquad y = \frac{c}{b} \qquad D = ac - b^2$$

The double autoresonance is forbidden in the blue region only.

Threshold and scaling



Numerical examples

How to find periodic trajectories in a system?

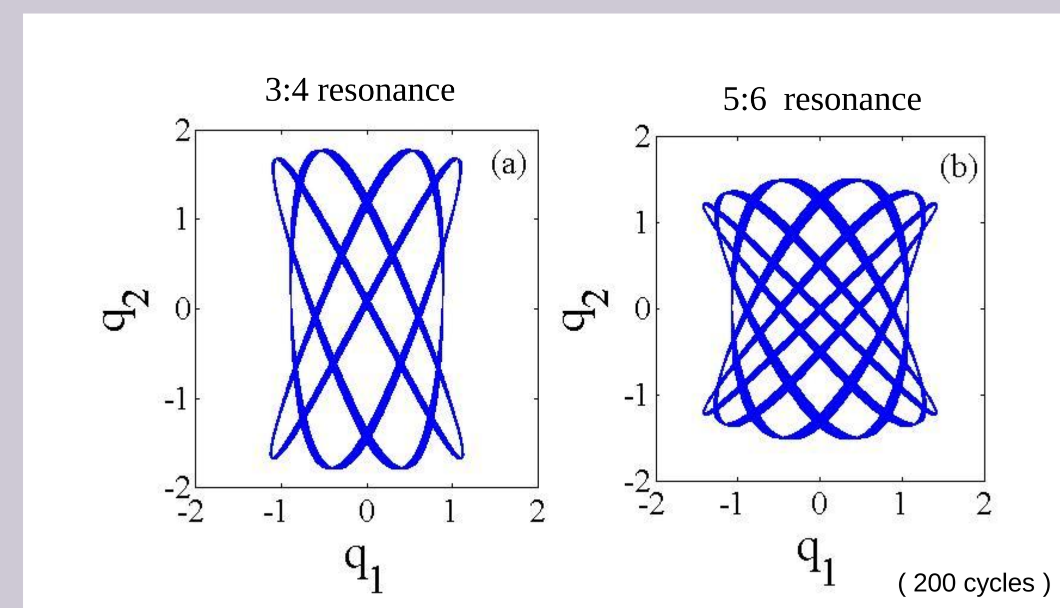
First, find the double autoresonance solution with a desire final rational ratio of the driving frequencies: $\frac{\omega_1}{\omega_2} = \frac{m}{n}$

The result is a nearly periodic orbit in the driven system as shown in examples A,B and C.

The next stage is to apply the iterative Newton's method with the AR solution as a first guess for the *undriven* system (example D).

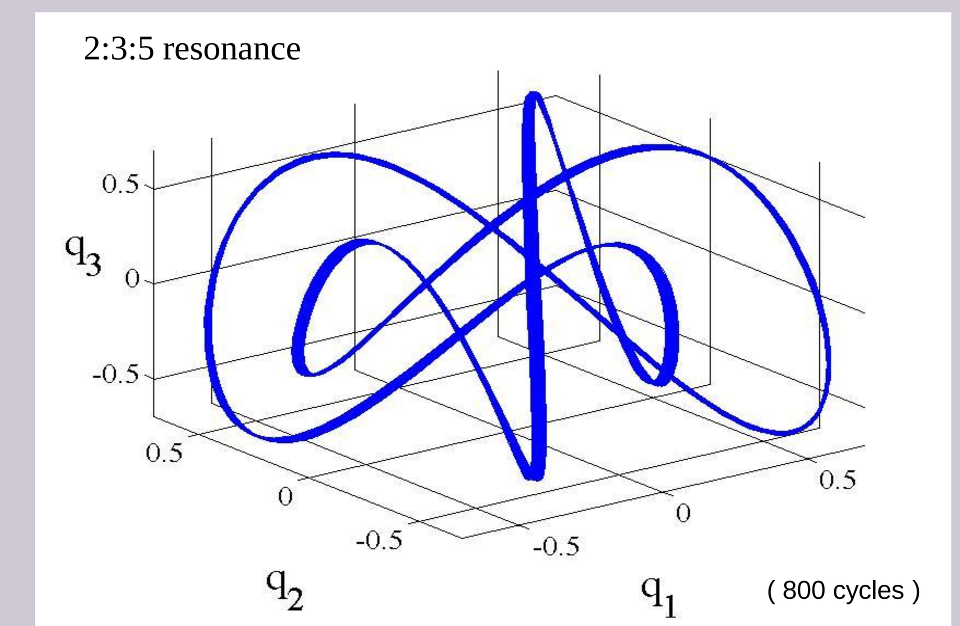
A. Quadric potential

$$V = \frac{\nu_1^2 q_1^2 + \nu_2^2 q_2^2}{2} + \gamma_1 q_1^4 + \gamma_2 q_2^4 + \delta q_1^2 q_2^2$$



B. Three coupled oscillators

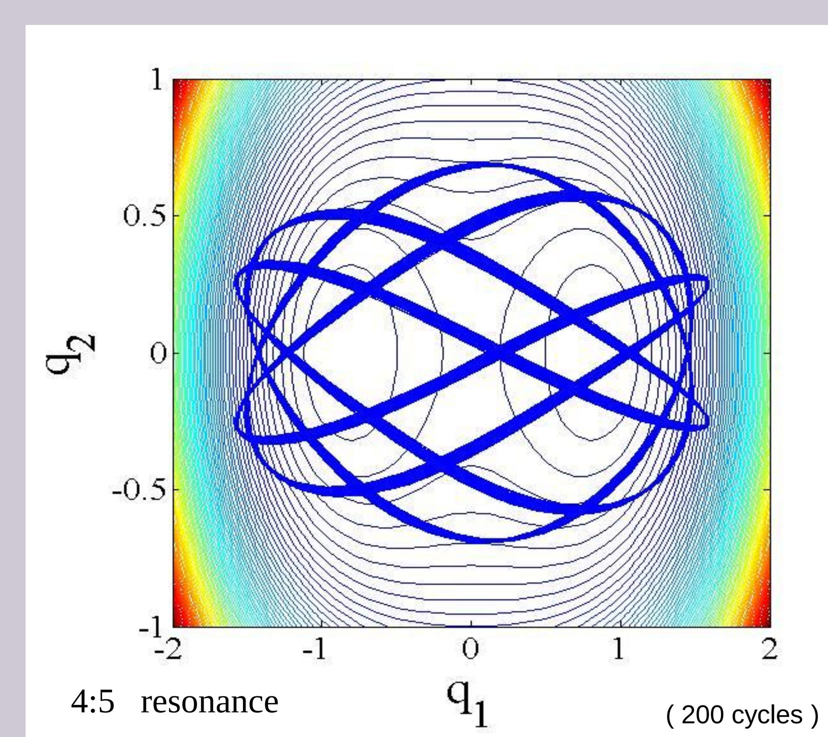
$$H_0 = \sum_{i=1}^{3}\left(\frac{p_i^2 + \nu_i^2 q_i^2}{2} + \gamma_i q_i^4\right) + \frac{\delta}{2}\sum_{i \neq j} q_i^2 q_j^2$$



C. Double-well potential

Combination of autoresonance and the *adiabatic switching* approaches.

$$V_{final} = \frac{-3\nu_1^2 q_1^2 + \nu_2^2 q_2^2}{2} + \left(q_1^2 + q_2^2\right)^2$$

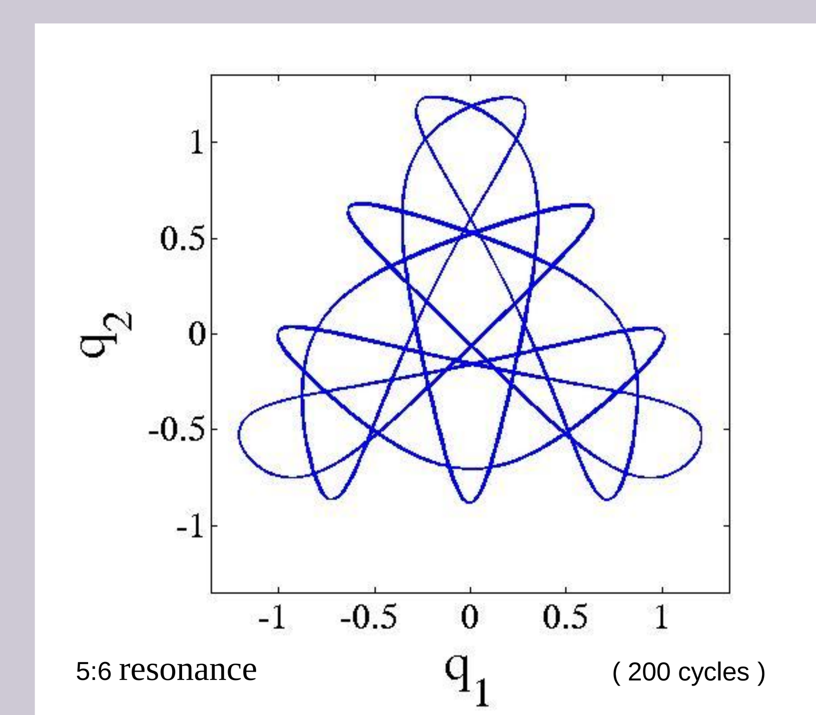


D. Toda Lattice (integrable)

$$V = \frac{1}{24}\left(e^{2q_2 + 2\sqrt{3}q_1} + e^{2q_2 - 2\sqrt{3}q_1} + e^{-4q_2}\right) - \frac{1}{8}$$

This potential is linearly degenerate $(\nu_1 = \nu_2)$ therefore we used special drive of type:

$$f = q_1\left(\varepsilon_1 \cos\varphi_1 + \varepsilon_2 \cos\varphi_2\right) + q_2\left(\varepsilon_1 \sin\varphi_1 - \varepsilon_2 \cos\varphi_2\right)$$



Conclusions

- Doubly autoresonant phase locking is stable despite slow variation of system's parameters.
- We classified possible stable autoresonant states in the system in the parameter space defined by the Hessian matrix of the unperturbed Hamiltonian in terms of action variables.
- The formation of nearly periodic trajectories in 2D systems is suggested as application of the AR phenomenon. This approach may be useful in semi-classical description of multi dimensional dynamical systems.
- The exact periodic orbits of an undriven system can be found from autoresonant, nearly periodic states By the Newton's method.
- We have tested our theory numerically for several systems.